Exercises
Learning and Intelligent Systems
SS 2017

Series 2, Mar 31, 2017
(Kernels)

It is not mandatory to submit solutions and sample solutions will be published in two weeks. If you choose to submit your solution, please send an e-mail from your ethz.ch address with subject Exercise2 containing a PDF (\LaTeX or scan) to harun.mustafa@inf.ethz.ch until Tuesday, Apr 11, 2017.

Problem 1 (Kernel Composition):
Assume that $k_i : \X \times \X \to \mathbb{R}$, $i = 1, \ldots, n$, are kernels with corresponding features mappings $\Phi_i : \X \to \mathbb{R}^{d_i}$. For each definition of $k(\cdot, \cdot)$ below, prove that $k$ is also a kernel by finding the corresponding mapping $\Phi : \X \to \mathbb{R}^d$.

(a) $k(x, y) := x^T M y$, for $x, y \in \mathbb{R}^d$, and some symmetric positive semidefinite matrix $M \in \mathbb{R}^{d \times d}$.

(b) $k(x, y) := \sum_{i=1}^n a_i k_i(x, y)$, for $a_1, \ldots, a_n > 0$. Hint: start by proving the fact for $n = 2$, then use mathematical induction.

(c) $k(x, y) := k_i(x, y)k_j(x, y)$

Problem 2 (Kernelized Linear Regression):
In this exercise you will derive the kernelized version of linear regression.

(a) Prove that the following identity holds for any matrix $B \in \mathbb{R}^{n \times m}$, and any invertible matrices $A \in \mathbb{R}^{m \times m}$, and $C \in \mathbb{R}^{n \times n}$.

$$(A^{-1} + B^T C^{-1} B)^{-1} B^T C^{-1} = AB^T (BAB^T + C)^{-1}$$

(b) Remember the solution of ridge regression, $w^* = (X^T X + \lambda I)^{-1} X^T y$. Use the matrix identity of part (a) to prove that $w^*$ lies in the row space of $X$, that is, it can be written as $w^* = X^T z^*$ for some $z^* \in \mathbb{R}^n$.

(c) Use the result of part (b) to transform the original ridge regression loss function,

$$R(w) = \|Xw - y\|^2_2 + \lambda \|w\|^2_2,$$

into a new loss function $\tilde{R}(z)$, such that $\tilde{R}(z^*) = R(w^*)$, and $z^* = \text{argmin}_z \tilde{R}(z)$.

(d) Assuming that you are given a kernel $k(\cdot, \cdot)$, express the kernel matrix $K$ of the data set as a function of the data matrix $X$, and substitute it in the new loss function $\tilde{R}(z)$ to obtain the kernelized version of the ridge regression loss function.

(e) To complete the kernelized version of ridge regression, show how you would predict the value $y$ of a new point $x$, assuming that you have already computed $z^*$. 
Problem 3 (Classifiers):
The following figure shows three classifiers trained on the same data set. One of them is a $k$-nearest neighbor classifier, and the other two are support vector machines (SVMs) using a quadratic and a Gaussian kernel respectively. Based on the shape of the decision boundary, can you guess which plot corresponds to which classifier?